Contracts for Density and Packing Functions

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August 1st, 2024

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- Hypergraph based "packing" reward functions.

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However, in contract theory problems we see only the outcome $f(S)$ and not the actions the agents take!

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- We see the reward $f(S)$, but not S.
- We transfer $t_i \cdot f(S)$ to each agent $i \in [n]$.
- So each agent $i \in [n]$ wants to take their action iff

$$\mathbb{E}_{\text{action}}\Big[t_i \cdot f(S \cup \{i\})\Big] - c_i \geq \mathbb{E}_{\text{nothing}}\Big[t_i \cdot f(S \setminus \{i\})\Big] - 0\,.$$

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So our optimal contract problem becomes finding a set $S \subseteq [n]$ that maximizes:

$$\left(1 - \sum_{i \in S} \frac{c_i}{f(S) - f(S \setminus \{i\})}\right) \cdot f(S)$$

Graph Supermodular Functions [DCVDPP24]

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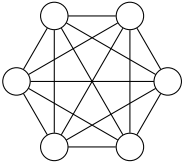

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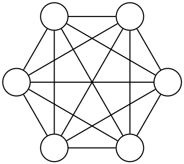

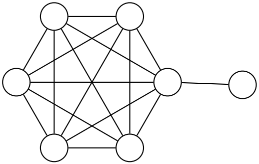

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In the identical costs special case there is no multiplicative approximation or additive FPTAS, unless P=NP. But, there is an additive PTAS.

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Open Question ([DCVDPP24])

Is there an additive PTAS in the general costs case?

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Theorem ([PS24])

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- Randomly round the LP.

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- We describe a new rounding procedure which:
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  - $deg_{S'}(v) = o(n)$ vertices will get concentration bounds.

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- Vertices in B might be included, and have degree $< \sigma \cdot n$ but could still have degree $\Omega(n/\log\log n)$.
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Our Approach in the General Cost Setting

Let's describe our approach more concretely.
Recall [DCVDPP24] partitioned V into $H, V \setminus H$ where

$$H = \{v \in V : \deg_{S'(v)} \geq \sigma \cdot n = \Omega(n)\}.$$

We partition V into A, B, C, D.

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- We obtain an (approximately) feasible optimal solution to the LP which can be randomly rounded to an approximately optimal solution for our original problem.

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  e.g. $f(S) = |E(S)|/\binom{n}{2}$ is an MPH-2 function.

Sanjeev Arora, David Karger, and Marek Karpinski.
Polynomial time approximation schemes for dense instances of np-hard problems.
In *Proceedings of the twenty-seventh annual ACM symposium on Theory of computing*, pages 284–293, 1995.

Ramiro Deo-Campo Vuong, Shaddin Dughmi, Neel Patel, and Aditya Prasad.
On supermodular contracts and dense subgraphs.
In *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 109–132. SIAM, 2024.

Paul Dütting, Tomer Ezra, Michal Feldman, and Thomas Kesselheim.
Multi-agent contracts.
In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*, pages 1311–1324, 2023.

Constantinos Daskalakis and Christos H Papadimitriou.
On oblivious ptas's for nash equilibrium.
In *Proceedings of the forty-first annual ACM symposium on Theory of computing*, pages 75–84, 2009.

Tomer Ezra, Michal Feldman, and Maya Schlesinger.
On the (in) approximability of combinatorial contracts.
In *15th Innovations in Theoretical Computer Science Conference (ITCS 2024)*. Schloss-Dagstuhl-Leibniz Zentrum für Informatik, 2024.